INJE–TP–95–7

Unstable two-dimensional extremal black holes

H.W. Lee and Y. S. Myung

Department of Physics, Inje University, Kimhae 621-749, Korea

Jin Young Kim

Division of Basic Science, Dongseo University, Pusan 616-010, Korea

arXiv:hep-th/9510125v1 18 Oct 1995

Abstract

We obtain the $\epsilon < 2$ new extremal ground states of a two-dimensional (2D) charged black hole where ϵ is the dilaton coupling parameter for the Maxwell term. The stability analysis is carried out for all these extremal black holes. It is found that the shape of potentials to an on-coming tachyon (as a spectator) take all barrier-well types. These provide the bound state solutions, which imply that they are unstable. We conclude that the 2D, $\epsilon < 2$ extremal black holes should not be considered as a toy model for the stable endpoint of the Hawking evaporation.

PACS number(s) : 04.70.Bw, 04.60.Kz, 11.25.Db

Recently the extremal black holes have received much attention. Extremal black holes provide a simple laboratory in which to investigate the quantum aspects of black hole [1]. One of the crucial features of these extremal black holes is that the Hawking temperature vanishes. The black hole with $M > Q$ will evaporate down to its extremal $M = Q$ state. Thus the extremal black hole may play the role of the stable endpoint for the Hawking evaporation. It has been also proposed that although the extremal black hole has nonzero area, it has zero entropy [2]. This is because the extremal case is distinct topologically from the nonextremal one.

It is very important to investigate the stability of the extremal black holes, which is essential to establish their physical existence. It has been shown that all 4D extremal charged black holes with the coupling parameter (a) are shown to be classically stable [3]. The $a = 0$ case corresponds to the Reissner-Nordström black hole. Since all potentials are positive definite outside of the outer horizon, one easily infer the stability of 4D extremal charged black holes using the same argument as employed by Chandrasekhar [4]. More recently, it is shown that the 2D extremal black holes are unstable [5].

In this Letter, we consider the two-dimensional dilaton gravity coupled to Maxwell and tachyon fields. The relevant coupling (parametrized by ϵ) between the dilaton and Maxwell field is included to obtain the general black hole. This may be considered as a two-dimensional counterpart of the 4D dilaton gravity with the parameter a. This coupling allows us to study the new $0 < \epsilon < 2$ and $\epsilon < 0$ extremal black holes. The $\epsilon = 0$ case corresponds to the 2D electrically charged extremal black hole. Our main task is to show whether these new extremal black holes are stable or not. As a compact criterion, the black bole is unstable if there exists a well-type potential to the on-coming physical waves [6,7]. This is so because, in solving the Schrödinger equation, the potential well always allows the bound state as well as scattering state. The former shows up as an imaginary frequency mode, leading to an exponentially growing mode. This indicates that the black hole is unstable.

We start with two-dimensional dilaton (Φ) gravity conformally coupled to Maxwell ($F_{\mu\nu}$) and tachyon (T) fields [8]

$$S = \int d^2x\sqrt{-G}e^{-2\Phi}\{R + 4(\nabla\Phi)^2 + \alpha^2 - \frac{1}{2}e^{\epsilon\Phi}F^2 - \frac{1}{2}(\nabla T)^2 + T^2\}. \tag{1}$$

The above action with $\epsilon = 0$ corresponds to the heterotic string. The tachyon is introduced to study the properties of black holes in a simple way. Setting $\alpha^2 = 8$ and after deriving equations of motion, we take the transformation

$$-2\Phi \to \Phi, \quad T \to \sqrt{2}T, \quad -R \to R. \tag{2}$$

Then the equations of motion become

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu\Phi + \nabla_\mu T\nabla_\nu T + \frac{4-\epsilon}{4}e^{-\epsilon\Phi/2}F_{\mu\rho}F_\nu{}^\rho = 0, \tag{3}$$

$$\nabla^2\Phi + (\nabla\Phi)^2 - \frac{1}{2}e^{-\epsilon\Phi/2}F^2 - 2T^2 - 8 = 0, \tag{4}$$

$$\nabla_\mu F^{\mu\nu} + \frac{2-\epsilon}{2}(\nabla_\mu\Phi)F^{\mu\nu} = 0, \tag{5}$$

$$\nabla^2 T + \nabla\Phi\cdot\nabla T + 2T = 0. \tag{6}$$

The general solution to the above equations is given by

$$\bar{\Phi} = 2\sqrt{2}r, \quad \bar{F}_{tr} = Q_E e^{-(2-\epsilon)\sqrt{2}r}, \quad \bar{T} = 0, \quad \bar{G}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix} \tag{7}$$

with

$$f = 1 - \frac{M}{\sqrt{2}}e^{-2\sqrt{2}r} + \frac{Q_E^2}{4(2-\epsilon)}e^{-(4-\epsilon)\sqrt{2}r}, \tag{8}$$

where M and Q_E are the mass and electric charge of the black hole, respectively. Note that from the requirement of $\bar{F}(r \to \infty) \to 0$ and $f(r \to \infty) \to 1$, we have the important constraint : $\epsilon < 2$. Hereafter we take $M = \sqrt{2}$ for convenience. In general, from $f = 0$ we can obtain two roots $(r_\pm)$ where $r_+(r_-)$ correspond to the event (Cauchy) horizons. The extremal black hole may provide a toy model to investigate the late stages of Hawking evaporation. We are mainly interested in the extremal limit (multiple root: $r_+ = r_- \equiv r_o$) of the charged black holes. The multiple root is obtained when $f(r_o) = 0$ and $f'(r_o) = 0$, in which case the square of charge is $Q_E^2 = 8(\frac{2-\epsilon}{4-\epsilon})^{(4-\epsilon)/2}$. Here the prime ($'$) denotes the derivative with respect to r. The extremal horizon is located at

$$r_o(\epsilon) = -\frac{1}{2\sqrt{2}}\log(\frac{4-\epsilon}{2-\epsilon}). \tag{9}$$

The explicit form of the extremal f is

$$f_e(r,\epsilon) = 1 - e^{-2\sqrt{2}r} + \frac{2}{(2-\epsilon)}(\frac{2-\epsilon}{4-\epsilon})^{(4-\epsilon)/2} e^{-(4-\epsilon)\sqrt{2}r}. \tag{10}$$

Fig. 1 shows the multiple roots of $f_e = 0$ at $r_o(\epsilon) = -1.076, -0.299$, and -0.119 for ϵ= 1.9, 0.5, and -3 respectively.

Now let us study whether these extremal ground states are stable or not. We introduce small perturbation fields around the background solution as [5]

$$F_{tr} = \bar{F}_{tr} + \mathcal{F}_{tr} = \bar{F}_{tr}[1 - \frac{\mathcal{F}(r,t)}{Q_E}], \tag{11}$$

$$\Phi = \bar{\Phi} + \phi(r,t), \tag{12}$$

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + h_{\mu\nu} = \bar{G}_{\mu\nu}[1 - h(r,t)], \tag{13}$$

$$T = \exp(-\frac{\bar{\Phi}}{2})[0 + t(r,t)]. \tag{14}$$

One has to linearize (3)-(6) in order to obtain the equations governing the perturbations. However, the stability should be based on the physical degrees of freedom. It is thus important to check whether the graviton (h), dilaton (ϕ), Maxwell mode ($\mathcal{F}$) and tachyon (t) are physically propagating modes in the 2D charged black hole background. We review the conventional counting of degrees of freedom. The number of degrees of freedom for the gravitational field ($h_{\mu\nu}$) in D-dimensions is $(1/2)D(D-3)$. For a Schwarzschild black hole, we obtain two degrees of freedom. These correspond to the Regge-Wheeler mode for odd-parity perturbation and Zerilli mode for even-parity perturbation [4]. We have -1 for $D = 2$. This means that in two dimensions the contribution of the graviton is equal and opposite to that of a spinless particle (dilaton). The graviton-dilaton modes $(h+\phi, h-\phi)$ are gauge degrees of freedom and thus turn out to be nonpropagating modes[6]. In addition, the Maxwell field has $D-2$ physical degrees of freedom. The Maxwell field has no physical degrees of freedom for $D = 2$. Since all these fields are nonpropagating modes, we will not consider equations (3)-(5). However, the tachyon as a spectator is a physically propagating

mode. This is introduced to illustrate many of the qualitative results about the 2D charged black hole in a simpler context. Its linearized equation is

$$f^2 t'' + f f' t' - [\sqrt{2} f f' - 2f(1-f)]t - \partial_t^2 t = 0. \tag{15}$$

To study the stability, we transform the above equation into one-dimensional Schrödinger equation. Introducing a tortoise coordinate

$$r \rightarrow r^* \equiv g(r),$$

(15) can be rewritten as

$$f^2 g'^2 \frac{\partial^2}{\partial r^{*2}} t + f\{f g'' + f' g'\} \frac{\partial}{\partial r^*} t - \{\sqrt{2} f f' - 2f(1-f)\}t - \frac{\partial^2}{\partial t^2} t = 0, \tag{16}$$

Requiring that the coefficient of the linear derivative vanish, one finds the relation

$$g' = \frac{1}{f}. \tag{17}$$

Assuming $t(r^*, t) \sim \tilde{t}(r^*) e^{i\omega t}$, one can cast (16) into the Schrödinger equation

$$\{\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\}\tilde{t} = 0, \tag{18}$$

where the effective potential $V(r)$ is given by

$$V(r) = f\{\sqrt{2} f' + 2(f-1)\}. \tag{19}$$

We are interested only in the extremal black holes. The potentials surrounding the extremal black holes are given by

$$V_e(r, \epsilon) = 2e^{-2\sqrt{2}r} f_e\{1 - 2(\frac{3-\epsilon}{2-\epsilon})(\frac{2-\epsilon}{4-\epsilon})^{(4-\epsilon)/2} e^{-(2-\epsilon)\sqrt{2}r}\}. \tag{20}$$

After a concrete analysis, one finds the barrier-well type potentials for $\epsilon < 2$. For examples, Fig. 2 shows the shapes of potentials for ϵ= 1.9, 0.5, and -3.

Now let us translate the potential $V_e(r, \epsilon)$ into $V_e(r^*, \epsilon)$. With f_e in (10), one finds the form of $r^* = \int^r \frac{dr}{f_e}$

$$r^* = r + \frac{1}{2\sqrt{2}(4-\epsilon)} \log|f_e| - \frac{2-\epsilon}{2\sqrt{2}(4-\epsilon)} \int^y \frac{dy}{1-y+Ay^{1+B}} \tag{21}$$

with $y = e^{-2\sqrt{2}r}, A = \frac{2}{(2-\epsilon)}[\frac{2-\epsilon}{4-\epsilon}]^{(4-\epsilon)/2}$, and $B = 1 - \frac{\epsilon}{2}$. Since both the forms of $V_e(r, \epsilon)$ and r^* are very complicated, we are far from obtaining the exact form of $V_e(r^*, \epsilon)$. At this point, it is crucial to obtain the approximate forms to $V_e(r^*, \epsilon)$ near the both ends. In the asymptotically flat region one finds from (21) that $r^* \simeq r$. (20) takes the asymptotic form

$$V_{r*\to\infty} \simeq 2\exp(-2\sqrt{2}r^*), \tag{22}$$

which is independent of ϵ. On the other hand, near the horizon ($r = r_o$) one has

$$r^* \simeq -\frac{2}{\sqrt{2}(2-\epsilon)} \frac{1}{(\frac{4-\epsilon}{2-\epsilon} - e^{-2\sqrt{2}r})}. \tag{23}$$

Approaching the horizon ($r \to r_o, r^* \to -\infty$), the potential takes the form

$$V_{r*\to-\infty} \simeq -\frac{1}{(4-\epsilon)} \frac{1}{r^{*2}}. \tag{24}$$

Using (22) and (24) one can construct the approximate form $V_{app}(r^*, \epsilon)$ (Fig. 3). This is also a barrier-well which is localized at the origin of r^*.

Our stability analysis is based on the equation

$$\{\frac{d^2}{dr^{*2}} + \omega^2 - V_{app}(r^*, \epsilon)\}\tilde{t} = 0. \tag{25}$$

As is well known, two kinds of solutions to the Schrödinger equation correspond to the bound and scattering states. In our case $V_{app}(r^*)$ admits two solutions depending on the signs of the energy ($E = \omega^2$): (i) For $E > 0(\omega = \text{real})$, the asymptotic solution for $\tilde{t}$ is given by $\tilde{t}_\infty = \exp(i\omega r^*) + R\exp(-i\omega r^*), r^* \to \infty$ and $\tilde{t}_{EH} = T\exp(i\omega r^*), r^* \to -\infty$. Here R and T are the scattering amplitudes of two waves which are reflected and transmitted by the potential $V_{app}(r^*, \epsilon)$, when a tachyonic wave of unit amplitude with the frequency ω is incident on the black hole from infinity. (ii) For $E < 0(\omega = -i\alpha$, α is positive and real), we have the bound state. Eq. (25) is given by

$$\frac{d^2}{dr^{*2}}\tilde{t} = (\alpha^2 + V_{app}(r^*))\tilde{t}. \tag{26}$$

The asymptotic solution is $\tilde{t}_\infty \sim \exp(\pm\alpha r^*), r^* \to \infty$ and $\tilde{t}_{EH} \sim \exp(\pm\alpha r^*), r^* \to -\infty$. To ensure that the perturbation falls off to zero for large r^*, we choose $\tilde{t}_\infty \sim \exp(-\alpha r^*)$. In the case of $\tilde{t}_{EH}$, the solution $\exp(\alpha r^*)$ goes to zero as $r^* \to -\infty$. Now let us observe whether or not $\tilde{t}_{EH} \sim \exp(\alpha r^*)$ can be matched to $\tilde{t}_\infty \sim \exp(-\alpha r^*)$. Assuming $\tilde{t}$ to be positive, the sign of $d^2\tilde{t}/dr^{*2}$ can be changed from + to − as r^* goes from ∞ to $-\infty$. If we are to connect $\tilde{t}_{EH}$ at one end to a decreasing solution $\tilde{t}_\infty$ at the other, there must be a point ($d^2\tilde{t}/dr^{*2} < 0$, $d\tilde{t}/dr^* = 0$) at which the signs of $\tilde{t}$ and $d^2\tilde{t}/dr^{*2}$ are opposite : this is compatible with the shape of $V_{app}(r^*, \epsilon)$ in Fig. 3. Thus it is possible for $\tilde{t}_{EH}$ to be connected to $\tilde{t}_\infty$ smoothly. Therefore a bound state solution is given by

$$\tilde{t}_\infty \sim \exp(-\alpha r^*), \qquad (r^* \to \infty) \tag{27}$$

$$\tilde{t}_{EH} \sim \exp(\alpha r^*), \qquad (r^* \to -\infty). \tag{28}$$

This is a regular solution everywhere in space at the initial time $t = 0$. It is well-known that in quantum mechanics, the bound state solution is always allowed if there exists a potential well. The time evolution of the solution with $\omega = -i\alpha$ implies $t_\infty(r^*, t) = \tilde{t}_\infty(r^*)\exp(-i\omega t) \sim \exp(-\alpha r^*)\exp(\alpha t)$ and $t_{EH}(r^*, t) = \tilde{t}_{EH}(r^*)\exp(-i\omega t) \sim \exp(\alpha r^*)\exp(\alpha t)$. This means that there exists an exponentially growing mode with time. Therefore, the $\epsilon < 2$ extremal ground states are classically unstable. The origin of this instability comes from the barrier-well potentials. These potentials appear from all $\epsilon < 2$ extremal black holes.

We conclude that the 2D, $\epsilon < 2$ extremal black holes cannot be considered as a toy model for the stable endpoint of the Hawking evaporation.

ACKNOWLEDGMENTS

This work was supported in part by Nondirected Research Fund, Korea Research Foundation,1994 and by Korea Science and Engineering Foundation, 94-1400-04-01-3.

REFERENCES

[1] For a review, see J. A. Harvey and A. Strominger, in *String theory and quantum gravity '92 :* Proc. the Trieste Spring School & Workshop (ICTP, March 1992) eds. J. Harvey, *et al* (World Scientific, Singapore, 1993).

[2] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Phys. Rev. **D51**, 4302(1995); C. Teitelbiom, *ibid.* **51**, 4315 (1995).

[3] C. Holzhey and F. Wilczek, Nucl. Phys. **B380**, 447 (1992).

[4] S. Chandrasekhar, *The Mathematical Theory of Black Hole* (Oxford Univ. Press, New York, 1983).

[5] J. Y. Kim and Y. S. Myung, Report No. INJE-TP-95-5, 1995; Y. S. Myung, Report No. INJE-TP-95-6, 1995.

[6] J.Y. Kim, H. W. Lee and Y. S. Myung, Phys. Lett. **B328**, 291 (1994) ; Y. S. Myung, Phys. Lett. **B334**, 29 (1994).

[7] E. Winstanley and E. Mavromatos, Phys. Lett. **B352**, 242 (1995).

[8] M. D. McGuigan, C. N. Nappi and S. A. Yost, Nucl. Phys. **B 375**, 421 (1992) ; O. Lechtenfeld and C. N. Nappi, Phys. Lett. **B288**, 72 (1992).

FIGURES

Fig. 1: Three graphs of extremal f_e for $\epsilon = 1.9$ (dashed line : —-—-), 0.5 (dotted line :- - - -), and -3 (solid line : —). The corresponding multiple roots (extremal horizons) are located at $r_o = -1.076, -0.299$, and -0.119 respectively.

Fig. 2: Three graphs of extremal potentials ($V_e(r, \epsilon)$) for $\epsilon = 1.9$ (dashed line : —-—-), 0.5 (dotted line :- - - -), and -3 (solid line : —). The potentials are zero at $r_o(\epsilon)$ and all barrier-well types outside $r_o(\epsilon)$.

Fig.3 : The approximate potential ($V_{app}(r^*, \epsilon)$). The aymptotically flat region is at $r^* = \infty$. This also takes a barrier-well type. This is localized at $r^* = 0$, falls to zero exponentially as $r^* \to \infty$ and inverse-squarely as $r^* \to -\infty$ (solid lines). The dotted line is used to connect two boundaries.